1 Series Solutions of Differential Equations

1.1 Introduction: The Taylor Polynomial Approximation

The best tool for numerically approximating a function $f(x)$ near a particular point x_0 is the Taylor polynomial.

The formula for the Taylor polynomial of degree n centered at x_0, approximating a function $f(x)$ possessing n derivatives x_0 is given by

$$p_n(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

Example

Find the first four Taylor polynomials for e^x, expanded around $x_0 = 0$.

$p_n(x)$ is written as $f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$.

Since we know the derivatives of $f(x) = e^x$ is just $e(x)$, $f^{(j)}(0) = 1$ for all of them.

This simplifies to $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.

The Taylor polynomial p_n is just the $(n+1)$st partial sum of the Taylor series

$$\sum_{j=0}^{\infty}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

Example

Determine the fourth-degree Taylor polynomials matching the function $\cos x$ at $x_0 = 2$

So using what was previously given we have $f(2) + f'(2)(x-2) = \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}}{4!}(x-2)^4$.

Filling in the $f^{(j)}$ values gives us $p_4(x) = \cos 2 - \sin 2(x-2) - \frac{\cos 2}{2}(x-2)^2 + \frac{\sin 2}{6}(x-2)^3 + \frac{\cos 2}{24}(x-2)^4$

Example

Find the first few Taylor polynomials approximating the solution around $x_0 = 0$ of the initiai value problem

$$y'' = 3y' + x^{7/2}y \qquad y(0) = 10 \quad y'(0) = 5$$

In general, this is just $y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(n)}(0)}{n!}x^n$.

Since we are given the problem, we know that $y''(0) = 3y'(0) + 0 = 15$.

As we continue taking derivatives with respect to x, we get $y''' = 3y'' + \frac{7}{3}x^{7/3}y + x^{7/3}y'$, and plugging in the numbrs gives us $y'''(0) = 45$.

Calculating the 4th derivative gives us 135.

The fifth derivative is no longer defined.

Example

Determine the Taylor polynomial of degree 3 for the solution to the initial value problem

$$y' = \frac{1}{x+y+1} \qquad y(0) = 0$$

Finding $y'(0)$ gives us 1, and finding $y''(0)$ gives us -2, and $y'''(0) = 10$.

We can estimate the accuracy to which a Taylor polynomial $p_n(x)$ approximates its target function $f(x)$ for x near x_0. The error $\epsilon_n(x)$ measures the accuracy of the approximation,

$$\epsilon_n(x) = f(x) - p_n(x)$$

and can be estimated by $\epsilon_n(x) = \frac{f^{(n+1)}(\aleph)}{(n+1)!}(x - x_0)^{n+1}$, where $\aleph$ is guaranteed to lie between x_0 and x if the $(n+1)$st derivative of f exists and is continuous on an interval containing x_0 and x.

1.2 Power Series and Analytic Functions

A power series about the point x_0 is an expression of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots$$

where x is a variable and the a_n's are constants.

A power series is convergent at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges, that is

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n(x - x_0)^n$$

If the limit does not exist at x, then the series is said to be divergent.

Every power series has an interval of convergence. The interval of convergence is the set of all real numbers x for which the series converges. The center of the interval of convergence is the center x_0 of the series. Within its interval of convergence a power series converges absolutely. In other words, if x is in the interval of convergence and is not an endpoint of the interval, then the series of absolute values

$$\sum_{n=0}^{\infty} |a_n(x - x_0)^n|$$

converges.

Theorem 1.1

For each power series, there is a number $\rho \quad (0 \leq \rho < \infty)$, called the radius of convergence of the power series, such that the series converges absolutely for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$. If the series converges for all values of x, then $\rho = \infty$. WHen the series converges only at x_0, then $\rho = 0$.

Theorem 1.2

If, for n large, the coefficients a_n are nonzero and satisfy

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = L \qquad (0 \leq L \leq \infty)$$

then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is $\rho = L$.

Example

Determine the interval and radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1} (x-3)^n$$

From the ratio test, the radius of convergence is $\rho = \frac{1}{2}$.

The interval of convergence is $|x - 3| < \frac{1}{2}$.

So the interval is $-5/2 < x < 7/2$.

For $7/2$, it converges, so $-5/2 < x \leq 7/2$.

Theorem 1.3

If $\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0$ for all x in some open interval, then each coefficient a_n equals zero.

Theorem 1.4

If the series $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has a positive radius of convergence ρ, then f is differentiable in the interval $|x - x_0| < \rho$ and termwise differentiation gives the power series for the derivative:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \qquad \text{for} \qquad |x - x_0| < \rho$$

Furthermore, termwise integration gives the power series for the integral of f:

$$\int f(x) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C \qquad \text{for} \qquad |x - x_0| < \rho$$

Example

Starting with the geometric series for $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$ find a power series for each of the following functions.

(a) $\frac{1}{1+x^2}$

Replace x with $-x^2$ and we get the power series equal to

$1 - x^2 + x^4 - x^6 + x^8 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad -1 < x < 1.$

(b) $\frac{1}{(1-x)^2}$

This becomes $1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$

(c) $\arctan x$ This becomes $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

Example

Express the series $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ as a series where the generic term is x^k instead of x^{n-2}.

Let $k = n - 2$, so $n = k + 2$.

Plugging this in gives us $\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k$.

Example

Show that $x^3 \sum_{n=0}^{\infty} n^2(n-2)a_n x^n = \sum_{n=3}^{\infty} (n-3)^2(n-5)a_{n-3}x^n$.

Let $k = n + 3$, so $n = k - 3$.

Doing stuff gives you the answer of $\sum_{n=3}^{\infty} (n-3)^2(n-5)a_{n-3}x^n$.

Exercise Show that the identity $\sum_{n=1}^{\infty} na_{n-1}x^{n-1} + \sum_{n=2}^{\infty} b_n x^{n+1} = 0$ implies that $a_0 = a_1 = a_2 = 0$ and $a_n = -\frac{b_{n-1}}{(n+1)}$ for $n \geq 3$.

Definition

A function f is said to be analytic at x_0 if, in an open interval about x_0, this function is the sum of a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ that has a positive radius of convergence.

A polynomial is analytic at every x_0. A rational function $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials without a common factor, is analytic except at those x_0 for which $Q(x_0) = 0$. The elementary functions $e^x, \sin x, \cos x$ are analytic for all x while $\ln x$ is analytic for $x > 0$. Familiar representations are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

where the first three are valid for all x, whereas the last is valid for x in $(0, 2]$.

1.3 Power Series Solutions to Linear Differential Equations

Definition

A point x_0 is called an ordinary point if both $p = a_1/a_2$ and $q = a_0/a_2$ are analytic at x_0. If x_0 is not an ordinary point, it is called a singular point of the equation.

Example

Determine all the singular points of

$$xy'' + x(1-x)^{-1}y' + (\sin x)y = 0$$

The form of this is $y'' + \frac{1}{1-x}y' + \frac{\sin x}{x}y = 0$.

$p(x) = \frac{1}{1-x}$ can be represented as a power series as well as $q(x) = \frac{\sin x}{x}$.

The only singular point is at $x = 1$.

Example

Find a power series solution about $x = 0$ to

$$y' + 2xy = 0$$

We are substituting around $y = \sum_{n=0}^{\infty} a_n x^n$. The derivative is $y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$.

Substituting this in gives $\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\sum} a_n x^n = 0$.

When we are trying to get x^1 in the summations, we get $a_1 + \sum_{n=2} na + nx^{n-1} + \sum_{n=0} 2a_n x^{n+1} = 0$.

Simplifying this gives us $a_1 + \sum_{k=1} [(k+1)a_{k+1} + 2a_{k-1}]x^k = 0$.

We have $a_{k+1} = \frac{-2a_{k-1}}{k+1}$.

From the expanded form of y we have $a_0 x_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$.

We already know $a_1 = 0$.

We can keep finding the formulas, $a_2 = \frac{-2}{2}a_0$, $a_4 = \frac{-2}{4} \cdot \frac{-2}{2}a_0$ and $a_6 = \frac{-2}{6} \cdot \frac{-2}{4} \cdot \frac{-2}{2}a_0$, and the odd k will result in 0.

We have $y = a_0 + \frac{-2}{2}a_0 x^2 + \frac{(-2)^2}{4 \cdot 2}a_0 x^4 + \frac{(-2)^3}{6 \cdot 4 \cdot 2}a_0 x^6 + \cdots + \frac{(-3)^n}{2 \cdot n!}x^{2n}$.

We can also write this as $y = a_0 \sum_{n=0} \frac{(-1)^n}{n!} x^{2n}$, which ends up being $a_0 e^{-x^2}$.

Example

Find a general solution to

$$2y'' + xy' + y = 0$$

in the form of a power series about the ordinary point $x = 0$.

We have $y'' + \frac{x}{2}y' + \frac{1}{2}y = 0$.

There are no singular points here, so all points are ordinary.

We will find this with $y = \sum_{n=0}^{\infty} a_n x^n$ and $y' = \sum_{n=1} a_n n x^{n-1}$ and $y'' = \sum_{n=2} a_n n(n-1)x^{n-2}$.

Plugging this in gives $2\sum_{n=2} a_n n(n-1)x^{n-2} + x\sum_{n=1} a_n n x^{n-1} + \sum_{n=0} a_n x^n = 0$.

This will simplify to $4a_2 + a_0 + \sum_{k=1} [2a_{k+2}(k+2)(k+1) + (k+1)a_k]x^k = 0$.

The recurrence formula ends up being $a_{k+2} = \frac{-a_k}{2(k+2)}$.

Let's look at $k = 1, k = 2, k = 3, k = 4$ until we find a pattern.

We also know $a_2 = -\frac{1}{4}a_0$.

We have that $a_3 = \frac{-a_1}{2\cdot 3}, a_4 = -\frac{a_2}{2\cdot 4}, a_5 = -\frac{a_3}{2\cdot 5}, a_6 = -\frac{a_4}{2\cdot 6}$.

We can write a_4 in terms of a_0 as $-\frac{1}{2\cdot 4} \cdot -\frac{1}{4}a_0$ and $a_6 = -\frac{2\cdot 6}{\cdot} - \frac{1}{2\cdot 4} \cdot -\frac{1}{4}a_0$.

With these patterns we can write this as $a_{2n+1} = \frac{(-1)^n}{2^n[(2n+1)\cdot \ldots 1]}$.

Ok we know $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a^4 x^4 + a^5 x^5 + \ldots$.

So we get this is equal to $a_0 + a_1 x - \frac{1}{4}a_0 x^2 - \frac{1}{6}a_1 x^3 + \frac{1}{32}a_0 x^4 + \frac{1}{60}a_1 x^5$.

This is a linear combination of a_0 and a_1.

Example

Find the first few terms in a power series expansion about $x = 0$ for a general solution to

$$(1 + x^2)y'' - y' + y = 0$$

Yea, a lot of stuff happen.

If you do previous steps of changing the indices and writing out the power series, we get

$$[2a_2 - a_1 + a_0] + [6a_3 - 2a_2 + a_1]x + \sum_{k=2} [(k+2)(k+1)a_{k+2} + (k+1)a_{k+1} + (k^2 - k + 1)a_k]x^k = 0$$

And then we can find $a_{k+2} = \frac{-(k+1)a_{k+1} - (k^2-k+1)a_k}{(k+2)(k+1)}$.

We also know $a_2 = \frac{1}{2}(a_1 - a_0)$ and $a_3 = \frac{1}{6}(2a_2 - a_1) =$ $frac{-}a_0 6$.

Doing many many steps gives you $y = a_0 + -\frac{1}{2}a_0 x^2 - \frac{1}{6}a_0 x^3 + \frac{1}{12}a_0 x^4 + \frac{3}{40}a_0 x^5 - \frac{17}{720}a_0 x^6$ for the case of when $a_1 = 0$.

When $a_0 = 0$, then the equation just becomes $a_1[x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \frac{1}{20}x^6 + \ldots]$.

1.4 Equations with Analytic Coefficients

We start by stating a basic existence theorem for the equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

Theorem 1.5

Suppose x_0 is an ordinary point for the equation. THen this equation has two linearly independent analytic solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Moreover, the radius of convergence of any power series solution of the form given by the above is at least as large as the distance from x_0 to the nearest singular point (real or complex-valued) of the original equation.

Example

Find a minimum value for the radius of convergence of a power series solution about $x = 0$ to

$$2y'' + xy' + y = 0$$

So we have $y'' + \frac{x}{2}y' + \frac{1}{2}y = 0$.

There are no singular points, so the radius of convergence is $\rho = \infty$

Example

Find a minimum value for the radius of convergence of a power series solution about $x = 0$ to

$$(1 + x^2)y'' - y' + y = 0$$

This is $y'' - \frac{1}{1+x^2}y' + \frac{1}{1+x^2}y = 0$.

The singular points are $\pm i$.

The distance from 0 is 1, so $\rho = 1$.

Example

Find the first few terms in a power series expansion about $x = 1$ for a general solution to

$$2y'' + xy' + y = 0$$

Also determine the radius of convergence of the series.

We can let $t = x - 1$, and $x = 1$ and $t = 0$.

So we can get $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$, so $Y(t) = y(x) = y(t+1)$.

We have $2\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} + (t+1)\frac{\mathrm{d}Y}{\mathrm{d}t} + Y = 0$.

Substituting some of this stuff in gives $2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + (t+1)\sum_{n=1}^{\infty} na_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$.

We need to break off some stuff, to simplify the sums.

We get $(4a_2 + a_1 + a_0)t^0 + \sum_{k=1} 2(k+2)(k+1)a_{k+2}t^k + \sum_{k=1} ka_k t^k + \sum_{k=1} (k+1)a_{k+1}t^k + \sum_{k=1} a_k t^k$.

We can get $a_{k+2} = \frac{-a_k - a_{k+1}}{2(k+2)}$.

We know of course that $Y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$.

We also know it's a linear combination, so $Y(t) = a_0(1 - \frac{1}{4}t^2 + \frac{1}{24}t^3 + \dots) + a_1(t - \frac{1}{4}t^2 - \frac{1}{8}t^3 + \dots)$

And just substitute $t = x - 1$ into the above to solve it.

1.5 Method of Frobenius

Definition

A singular point x_0 of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

is said to be a regular singular point if both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0. Otherwise x_0 is called an irregular singular point.

Example

Classify the singular points of the equation

$$(x^2 - 1)^2 y''(x) + (x+1)y'(x) - y(X) = 0$$

Rewriting this gives you $y'' + \frac{(x+1)}{(x+1)^2(x-1)^2}y' - \frac{1}{(x+1)^2(x-1)^2}y = 0$.

The singular points are $x = 1$ and $x = -1$.

$x = 1$ is an irregular singular point because it is not analytic for both $p(x)$ and $q(x)$. -1 is a regular singular point.

Definition

If x_0 is a regular singular point of $y'' + py' + qy = 0$, then the indical equation for this point is

$$r(r-1) + p_0 r + q_0 = 0$$

where

$$p_0 := \lim_{x \to x_0} (x - x_0)p(x), \qquad q_0 := \lim_{x \to x_0} (x - x_0)^2 q(x)$$

The roots of the indicial equation are called the exponents (indices) of the singularity x_0.

Example

Find the indical equation and the exponents of the singularity $x = -1$ of

$$(x^2 - 1)^2 y''(x) + (x+1)y'(x) - y(x) = 0$$

In standard form we have $y'' + \frac{(x+1)}{(x+1)^2(x-1)^2}y' - \frac{1}{(x+1)^2(x-1)^2}y = 0$.

We have $(x+1)p(x) = \frac{1}{(x-1)^2}$ and $(x+1)^2 q(x) = \frac{-1}{(x-1)^2}$.

The limits are $1/4$ and $-1/4$ respectively from this.

So the indical equation becomes $r(r-1) + p_0 r + q_0 = 0$ or $r(r-1) + \frac{1}{4}r - \frac{1}{4} = 0$ or $r^2 - \frac{3}{4}r - \frac{1}{4} = 0$

Factoring gives $(4r+1)(r-1)$, and the indical roots are $r = -1/4$ and $r = 1$.

Example

Find a series expansion about the regular singular point $x = 0$ for a solution to

$$(x+2)x^2 y''(x) - xy'(x) + (1+x)y(x) = 0, \qquad x > 0$$

Finding the indical roots gives us $p_0 = -1/2$, and $q_0 = 1/2$.

The indicial equation is $2r^2 - 3r + 1 = 0$, so the indicial roots are $r = 1/2$ and $r = 1$.

Now expand about $r = 1$.

We get $(x+2)x^2 \sum a_n(n+1)nx^{n-1} - x \sum a_n(n+1)x^n + (1+x)\sum a_n x^{n+1} = 0$.

Do some simplification to get

$\sum n = 0 a_n(n+1)nx^{n+2} + \sum_{n=1} 2a_n(n+1)nx^{n+1} - \sum_{n=1} a_n(n+1)x^{n+1} \sum_{n=0} a_n x^{n+2}$.

Writing them to start all at the same index and combining gives you $\sum_{k=2} [a_{k-2}(k-1)(k-2) + 2a_{k-1}k(k-1) - a_{k-1}(k-1) + a_{k-2}]x^k = 0$.

Finding the recurrence formula gives $a_{k-1} = \frac{-(k^2-3k+3)}{(2k-1)(k-1)}a_{k-2}$.

Putting k values into the formula gives you $y = x - \frac{1}{3}x^2 + \frac{1}{10}x^3 - \frac{1}{30}x^4 + \ldots$.

Theorem 1.6

If x_0 is a regular singular point, then there exists at least one series solution, where $r = r_1$ is the larger root of the associated indicial equation. Moreoever, this series converges for all x such that $0 < x - x_0 < R$, where R is the distance from x_0 to the nearest other singular point (real or complex).

Example

Find a series solution about the regular singular point $x = 0$ of

$$x^2y''(x) - xy'(x) + (1 - x)y(x) = 0, \qquad x > 0$$

We have $x = 0$ is a regular singular point from writing this in general form.

Writing the indicial equation gives us $r = 1$.

Writing the summations gives you $\sum\limits_{n=0} a_n(n+1)nx^{n+1} - \sum\limits_{n=0} a_n(n+1)x^{n+1} + \sum\limits_{n=0} a_n x^{n+1} - \sum\limits_{n=0} a_n x^{n+2} = 0$.

Simplify this to get $a_{k-1} = \frac{a_{k-2}}{(k=1)^2}$.

You end up getting $y = x + x^2 + \frac{1}{4}x^3 + \frac{1}{36}x^4 + \dots$.